TRANSACTIONS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 336, Number 2, April 1993

FUNCTIONAL EQUATIONS SATISFIED BY INTERTWINING OPERATORS OF REDUCTIVE GROUPS

CHEN-BO ZHU

ABSTRACT. This paper generalizes a recent work of Vogan and Wallach [VW] in which they derived a difference equation satisfied by intertwining operators of reductive groups. We show that, associated with each irreducible finite-dimensional representation, there is a functional equation relating intertwining operators. In this way, we obtain natural relations between intertwining operators for different series of induced representations.

0. INTRODUCTION

We use the convention of denoting a Lie group by a capital letter, and denoting its Lie algebra by the corresponding lower case German letter. A subscript $\mathbb{C}$ denotes complexification.

Let G be a real reductive group, $P = MAN$ a parabolic subgroup of G with a given Langlands decomposition [Kn], and $\Phi(P, A)$ the set of positive restricted A-roots corresponding to N. For $\nu \in \mathfrak{a}_{\mathbb{C}}^*$, let a^ν be the character of A: $a^\nu = e^{\nu(\log a)}$, $a \in A$. Let σ be an admissible representation of M, and H_σ the representation space. In the sequel, we shall require σ to have an infinitesimal character, whose definition is given in (2.1.5). We denote by $I_{P,\sigma,\nu} = \mathrm{Ind}_{MAN}^G(\sigma \otimes a^\nu \otimes 1)$ the space of C^∞ functions, f, from G to H_σ such that $f(xman) = a^{-(\nu+\rho)}\sigma^{-1}(m)f(x)$, where $m \in M$, $a \in A$, $n \in N$, and $\rho = \rho_P$, the half sum of the positive restricted A-roots counted with multiplicities. G acts on $I_{P,\sigma,\nu}$ by left translation. $I_{P,\sigma,\nu}$ is usually referred to as the generalized principal series.

Let $\overline{P} = MA\overline{N}$ be the opposite parabolic subgroup to P. We also define a representation of G by left translation in $I_{\overline{P},\sigma,\nu} = \mathrm{Ind}_{MA\overline{N}}^G(\sigma \otimes a^\nu \otimes 1)$, the space of C^∞ functions, f, from G to H_σ such that $f(xma\overline{n}) = a^{-(\nu-\rho)}\sigma^{-1}(m)f(x)$. We recall the standard (but formal) intertwining operator:

$$J(\overline{P} : P, \sigma, \nu)f(x) = \int_{\overline{N}} f(x\overline{n})\, d\overline{n}, \qquad f \in I_{P,\sigma,\nu}.$$

It is well known (see [Kn]) that if $\mathrm{Re}(\nu, \alpha) \geq c = c_\sigma$ for $\alpha \in \Phi(P, A)$ and c_σ some constant depending only on σ, than the integral defining

Received by the editors January 25, 1991.
1991 *Mathematics Subject Classification*. Primary 22E30, 22E46; Secondary 15A69.
Key words and phrases. Reductive groups, intertwining operators, functional equations.

$J(\overline{P}:P,\sigma,\nu)$ converges absolutely. Moreover, in the range of convergence, we have $J(\overline{P}:P,\sigma,\nu): I_{P,\sigma,\nu} \to I_{\overline{P},\sigma,\nu}$ and it intertwines the G-actions.

The problem of meromorphically continuing the operators $J(\overline{P}:P,\sigma,\nu)$ was resolved in the early seventies by purely analytic methods (see [KS1, KS2]). It was shown, among other things, that in the rank one case, one can in fact analytically continue these operators with respect to the parameter ν, except at negative multiples of some value of ν where they have simple poles. In this sense, these intertwining operators behave typically like the classical gamma function.

Therefore, it seems tempting and natural to ask whether these intertwining operators indeed have functional equations like the classical gamma function. This was first shown to be so by a recent work of Vogan and Wallach [VW] in which they derived a difference equation satisfied by intertwining operators. The method they employed was by tensoring with a finite-dimensional spherical representation. Clearly, that is the most efficient way of establishing the meromorphic continuation.

The purpose of the present work is to generalize and, at the same time, to explain the Vogan-Wallach result by tensoring with arbitrary finite-dimensional representations. It is fair to say that our main contribution is to show how simple and general the result turns out to be.

Some words about the organization of this paper are in order. In §1 (§§1.1–1.2), some very general constructions from multilinear algebra are discussed. These constructions will enable us to define four maps T, S, U, V and obtain most of their properties in a rather transparent way.

In §2 (§§2.1–2.8), two commuting diagrams are proved. The first commuting diagram (§2.4) relates T, S with intertwining operators. Its proof relies solely on a simple property of some projection operator, which is discussed in §2.2. Our main result is contained in the second commuting diagram (§2.7), which relates U, V with intertwining operators. Its proof uses a critical lemma due to Vogan, the first commuting diagram, and some general properties of T, S, U, V. What is important here is that the second commuting diagram gives us a functional equation connecting the intertwining operators $J(\mu,\nu)$ and $J(\mu+\mu_1,\nu+\nu_1)$, where (μ_1,ν_1) is a regular dominant integral infinitesimal character coming from an irreducible finite-dimensional representation of G. Moreover, it can be used to express $J(\mu,\nu)$ in terms of $J(\mu+\mu_1,\nu+\nu_1)$, not the other way around as in the case of the first commuting diagram. Following [VW], the last section §2.8 gives some basic properties of the factor $r_\lambda^G(\Lambda)$, which occurs in the second commuting diagram.

This work is part of the author's Yale thesis. He would like to thank his advisor, Roger Howe, for initiating the project and providing invaluable guidance throughout his graduate study.

1. Some multilinear algebra

1.1. **Generalities on dual and tensor product of vector spaces.** Let F be a *finite-dimensional* complex vector space, and F^* its complex dual. In this section, all our maps are linear.

There is a natural map

(1.1.1) $$\operatorname{tr}: F \otimes F^* \to \mathbb{C}$$

specified by $\operatorname{tr}(v \otimes v^*) = v^*(v)$, $v \in F$, $v^* \in F^*$. This induces maps

(1.1.2) $$U \otimes F \otimes F^* \xrightarrow{1_U \otimes \operatorname{tr}} U, \qquad (1_U \otimes \operatorname{tr})(u \otimes v \otimes v^*) = v^*(v)u, \qquad u \in U.$$

Here, U is a complex vector space, possibly infinite dimensional.
We can also think of this as defining *contraction* maps

(1.1.3) $$\circ v^*: \ U \otimes F \to U, \qquad a \circ v^* = (1_U \otimes \operatorname{tr})(a \otimes v^*), \qquad a \in U \otimes F.$$

Thus, $(u \otimes v) \circ v^* = v^*(v)u$, $u \in U$.

(1.1.4) *Remark.* For any dual bases $\{y_i, y_i^*\}$ of F and F^*, we have

$$\sum_i (a \circ y_i^*) \otimes y_i = a, \qquad a \in U \otimes F.$$

(1.1.5) *Remark.* The contraction mapping has the following property:

$$(1_U \otimes A)(a) \circ v^* = a \circ A^*(v^*), \qquad a \in U \otimes V, \ A \in \operatorname{End}(F),$$

where $A^* \in \operatorname{End}(F^*)$ is the adjoint of A.
This may be checked by the formula

$$\begin{aligned}(1_U \otimes A)(u \otimes v) \circ v^* &= (u \otimes Av) \circ v^* = v^*(Av)u \\ &= (A^*v^*)(v)u = (u \otimes v) \circ A^*(v^*).\end{aligned}$$

(1.1.6) *Remark.* Let $F_1 \subseteq F$ be a subspace. A map $P_1 \in \operatorname{End}(F)$ is called a projection operator from F onto F_1 if the image of P_1 is F_1 and $P_1|_{F_1} = 1_{F_1}$. By applying $1_U \otimes P_1$ to the equation in Remark (1.1.4) and using Remark (1.1.5), we see

$$\sum_i (a \circ P_1^*(y_i^*)) \otimes P_1(y_i) = a, \qquad a \in U \otimes F_1.$$

Given a *map*

(1.1.7) $$T: \ X \to U \otimes F,$$

there is an *associated map*

(1.1.8) $$T^\dagger: \ X \otimes F^* \to U$$

defined by the following composition:

$$X \otimes F^* \xrightarrow{T \otimes 1_{F^*}} U \otimes F \otimes F^* \xrightarrow{1_U \otimes \operatorname{tr}} U.$$

Thus, $T^\dagger(x \otimes v^*) = T(x) \circ v^*$.

We can *reconstruct* T *from* $T^\dagger$ as follows: Given $S: \ X \otimes F^* \to U$, we can define $S^\dagger: \ X \to U \otimes F$ via the diagram:

(1.1.9) $$X \xrightarrow{j} X \otimes F^* \otimes F \xrightarrow{S \otimes 1_F} U \otimes F,$$

where $j(x) = \sum_i x \otimes y_i^* \otimes y_i$, with $\{y_i\}$ any basis of F and $\{y_i^*\}$ the dual basis of F^*. Thus,

$$S^\dagger(x) = \sum_i S(x \otimes y_i^*) \otimes y_i. \tag{1.1.10}$$

(1.1.11) *Remark.* The expression $\sum_i y_i^* \otimes y_i \in F^* \otimes F$ is independent of the choice of the dual bases $\{y_i\}$ and $\{y_i^*\}$. Under the well-known identification $F^* \otimes F \cong \mathrm{End}(F)$, $\sum_i y_i^* \otimes y_i \in F^* \otimes F$ corresponds to the identity element of $\mathrm{End}(F)$. Here it is worthwhile to point out the following general principle:

For any bilinear map $B(\cdot\,,\cdot) : F^* \times F \to W$, the expression $\sum_i B(y_i^*, y_i) \in W$ is independent of the choice of the dual bases $\{y_i\}$ and $\{y_i^*\}$.

Now if we have $T : X \to U \otimes F$, we then have $T^\dagger : X \otimes F^* \to U$ and $(T^\dagger)^\dagger : X \to U \otimes F$.

We compute

$$(T^\dagger)^\dagger(x) = \sum_i T^\dagger(x \otimes y_i^*) \otimes y_i = \sum_i (T(x) \circ y_i^*) \otimes y_i = T(x)$$

(by Remark (1.1.4)). Hence, $(T^\dagger)^\dagger = T$.

1.2. **Generalities on induction, dual, tensor product of representations.** Let G be a group and H a subgroup. Let ρ be a *finite-dimensional* representation of G, and μ a representation of H, possibly infinite-dimensional. Then

$$\rho : G \to \mathrm{Gl}(F), \qquad \mu : H \to \mathrm{Gl}(U).$$

We define λ_μ to be the representation of G induced from μ, i.e.,

$$\lambda_\mu = \mathrm{ind}_H^G \mu = \{f : G \to U | f(gh) = \mu^{-1}(h) f(g), \quad g \in G, h \in H\}.$$

G acts on λ_μ by left translation.

Since ρ is a finite-dimensional representation, tensoring with ρ is a well-defined, purely algebraic operation.

The following proposition is quite routine, but because it is simple and yet very useful for our later purpose, we include a detailed proof.

(1.2.1) **Proposition.** *The mapping* $\alpha : \lambda_\mu \otimes \rho \to \lambda_{\mu \otimes \rho|_H}$ *specified by*

$$\alpha(f \otimes v)(g) = f(g) \otimes \rho(g)^{-1} v, \qquad f \in \lambda_\mu, \ v \in F, \ g \in G,$$

is a G*-isomorphism.*

Proof. α is a G-map, since

$$\begin{aligned}
\alpha\left((\lambda_\mu \otimes \rho)(g')\left(\sum_i f_i \otimes v_i\right)\right)(g) &= \alpha\left(\sum_i \lambda_\mu(g') f_i \otimes \rho(g') v_i\right)(g) \\
&= \sum_i (\lambda_\mu(g') f_i)(g) \otimes \rho(g)^{-1} \rho(g') v_i \\
&= \sum_i f_i(g'^{-1} g) \otimes \rho((g'^{-1} g)^{-1} v_i) \\
&= \alpha\left(\sum_i f_i \otimes v_i)(g'^{-1} g\right)
\end{aligned}$$

and

$$\begin{aligned}\alpha\left(\sum f_i\otimes v_i\right)(gh) &= \sum_i f_i(gh)\otimes\rho((gh)^{-1})v_i\\ &= \sum \mu(h)^{-1}f_i(g)\otimes\rho(h)^{-1}\rho(g^{-1})v_i\\ &= (\mu\otimes\rho|_H)(h)^{-1}\left(\sum_i f_i(g)\otimes\rho(g)^{-1}v_i\right)\\ &= (\mu\otimes\rho|_H)(h)^{-1}\left(\alpha\left(\sum f_i\otimes v_i\right)(g)\right).\end{aligned}$$

We construct an *inverse of* α. Let $\phi\in\lambda_{\mu\otimes\rho|_H}$; then $\phi(g)\in U\otimes F$. Given a basis $\{y_i\}$ of F and a dual basis $\{y_i^*\}$ of F^*, we know that $\{\rho(g)^{-1}y_i\}$ and $\{\rho^*(g)^{-1}y_i^*\}$ are again dual bases of F and F^*. Therefore, we can write

$$\phi(g)=\sum_i(\phi(g)\circ\rho^*(g)^{-1}y_i^*)\otimes\rho(g)^{-1}y_i \quad \text{(see Remark (1.1.4)).}$$

Let

(1.2.2) $$\beta\phi=\sum\phi_i\otimes y_i,$$

where $\phi_i(g)=\phi(g)\circ\rho^*(g)^{-1}y_i^*$.

We check

$$\begin{aligned}(\alpha\beta)(\phi)(g)&=\alpha\left(\sum\phi_i\otimes y_i\right)(g)=\sum\phi_i(g)\otimes\rho(g)^{-1}y_i\\ &=\sum_i(\phi(g)\circ\rho^*(g)^{-1}y_i^*)\otimes\rho(g)^{-1}y_i=\phi(g).\end{aligned}$$

$$(\beta\alpha)(f\otimes v)=\sum f_i\otimes y_i,$$

where

$$\begin{aligned}f_i(g)&=\alpha(f\otimes v)(g)\circ\rho^*(g)^{-1}y_i^*=(f(g)\otimes\rho(g)^{-1}v)\circ\rho^*(g)^{-1}y_i^*\\ &=f(g)\cdot(\rho^*(g)^{-1}y_i^*)(\rho(g)^{-1}v)=f(g)\cdot y_i^*(v),\end{aligned}$$

so

$$(\beta\alpha)(f\otimes v)=\sum f\cdot y_i^*(v)\otimes y_i=f\otimes v.$$

Now suppose $\nu: H\to \mathrm{Gl}(X)$ is another representation of H, and suppose we have a *H-intertwining map*

(1.2.3) $$\begin{aligned}T_0\colon X&\to U\otimes F\\ \nu&\to\mu\otimes\rho|_H.\end{aligned}$$

This is equivalent to having a *H-intertwining map*

(1.2.4) $$T_0^\dagger:\ X\otimes F^*\to U$$

defined by $T_0^\dagger(x\otimes v^*)=T_0(x)\circ v^*$.

The map T_0 induces a map

(1.2.5) $$(T_0)_G:\ \lambda_\nu\to\lambda_{\mu\otimes\rho|_H}$$

by simply composing function values:

$$(T_0)_G(f)(g) = T_0(f(g)), \qquad f \in \lambda_\nu.$$

Composing with $\beta : \lambda_{\mu\otimes\rho|_H} \to \lambda_\mu \otimes \rho$ gives us a G-intertwining map

$$\beta(T_0)_G : \lambda_\nu \xrightarrow{(T_0)_G} \lambda_{\mu\otimes\rho|_H} \xrightarrow{\beta} \lambda_\mu \otimes \rho \tag{1.2.6}$$

which is computed by the formula

$$(\beta(T_0)_G)(f) = \sum_i f_i \otimes y_i, \tag{1.2.7}$$

where $f_i(g) = T_0(f(g)) \circ \rho^*(g)^{-1} y_i^*$, $\{y_i\}$ is, as above, a basis for F, and $\{y_i^*\}$ is the dual basis of F^*.

The map $T_0^\dagger : X \otimes V^* \to U$ induces a map

$$(T_0^\dagger)_G : \lambda_{\nu\otimes\rho^*|_H} \to \lambda_\mu \tag{1.2.8}$$

again by composing function values:

$$(T_0^\dagger)_G(\phi)(g) = (T_0^\dagger)(\phi(g)), \qquad \phi \in \lambda_{\nu\otimes\rho^*|_H}.$$

Composing with $\alpha : \lambda_\nu \otimes \rho^* \to \lambda_{\nu\otimes\rho^*|H}$ gives us a G-intertwining map

$$(T_0^\dagger)_G\alpha : \lambda_\nu \otimes \rho^* \xrightarrow{\alpha} \lambda_{\nu\otimes\rho^*|H} \xrightarrow{(T_0^\dagger)_G} \lambda_\mu, \tag{1.2.9}$$

which is computed by the formula

$$((T_0^\dagger)_G\alpha)(f \otimes v^*)(g) = T_0(f(g)) \circ \rho^*(g)^{-1} v^*, \qquad f \in \lambda_\nu. \tag{1.2.10}$$

Let us compare (1.2.6) with (1.2.9). If $f \in \lambda_\nu$, and $v^* \in F^*$, then we compute

$$(\beta(T_0)_G)^\dagger(f \otimes v^*) = (\beta(T_0)_G)f \circ v^* = \left(\sum f_i \otimes y_i\right) \circ v^* = \sum f_i \cdot v^*(y_i),$$

where

$$f_i(g) = T_0(f(g)) \circ \rho^*(g)^{-1} y_i^*.$$

Therefore,

$$\begin{aligned} ((\beta(T_0)_G)^\dagger)(f \otimes v^*)(g) &= T_0(f(g)) \circ \sum v^*(y_i)\rho^*(g)^{-1} y_i^* \\ &= T_0(f(g)) \circ \rho^*(g)^{-1} v^*. \end{aligned}$$

In other words,

$$(\beta(T_0)_G)^\dagger = (T_0^\dagger)_G\alpha. \tag{1.2.11}$$

Suppose we have a *submodule*

$$\rho_1 \subseteq \rho|_H, \qquad F_1 \subseteq F. \tag{1.2.12}$$

Let P_1 be a projection map from F onto F_1 (not necessarily a H-map) (see Remark (1.1.6) for the definition of such a map). Suppose the image of $T_0 \subseteq U \otimes F_1$. Then

$$T_0 = (1_U \otimes P_1)T_0.$$

Hence if $f \in \lambda_\nu$ and $v^* \in F^*$, we can compute

$$
\begin{aligned}
(\beta(T_0)_G)^\dagger(f \otimes v^*)(g) &= (T_0^\dagger)_G\alpha(f \otimes v^*)(g) \\
&= T_0(f(g)) \circ \rho^*(g)^{-1}v^* \\
&= (1_U \otimes P_1)T_0(f(g)) \circ \rho^*(g)^{-1}v^* \\
&= T_0(f(g)) \circ P_1^*(\rho^*(g)^{-1}v^*) \quad \text{(by Remark (1.1.5))}.
\end{aligned} \tag{1.2.13}
$$

Therefore, we have

$$
(\beta(T_0)_G)(f) = \sum f_i \otimes y_i, \qquad f \in \lambda_\nu, \tag{1.2.14}
$$

where $f_i = (\beta(T_0)_G)^\dagger(f \otimes y_i^*) = T_0(f(g)) \circ P_1^*(\rho^*(g)^{-1}y_i^*)$.

2. Natural relations between intertwining operator

2.1. **Preliminaries.** Let $\mathfrak{g}$ be a complex reductive Lie algebra. Choose a Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$, and thus a Cartan subalgebra $\mathfrak{h} \subseteq \mathfrak{b}$.

(2.1.1) The irreducible finite-dimensional representations of $\mathfrak{g}$ can be parametrized by their highest weights with respect to $\mathfrak{b}$, which are dominant integral (see [Hu]).

(2.1.2) Let $\mathscr{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$, and $\mathscr{Z}(\mathfrak{g})$ its center. We know from Harish-Chandra (see [V1], for example) that the homomorphisms of $\mathscr{Z}(\mathfrak{g})$ into $\mathbb{C}$ can be parametrized by elements of $\mathfrak{h}^*/W_G$, the set of Weyl group orbits in $\mathfrak{h}^*$. Let C_G be the positive Weyl chamber specified by our choice of $\mathfrak{b}$. It is a fundamental domain for the action of W_G on $\mathfrak{h}^*$. For $\lambda \in C_G \subseteq \mathfrak{h}^*$, the corresponding homomorphism of $\mathscr{Z}(\mathfrak{g})$ into $\mathbb{C}$ is denoted by χ_λ.

We fix our parabolic subgroup $P = MAN$ as in the Introduction.

Choose a Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{m}_\mathbb{C}$, and thus a Cartan subalgebra $\mathfrak{h}_0 \subseteq \mathfrak{b}_0$. Then $\mathfrak{b} = \mathfrak{b}_0 \oplus \mathfrak{n}_\mathbb{C}$ is a Borel subalgebra of $\mathfrak{g}_\mathbb{C}$, and $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{a}_\mathbb{C}$ is a Cartan subalgebra contained in $\mathfrak{b}$.

(2.1.3) We can apply (2.1.1) to the Lie algebras $\mathfrak{m}_\mathbb{C}$ and $\mathfrak{g}_\mathbb{C}$. Thus if $\mu_1 \in \mathfrak{h}_0^*$ is dominant integral with respect to $\mathfrak{b}_0$ and in the weight lattice, the unique irreducible finite-dimensional representation of M with this highest weight is denoted by σ_{μ_1}. Similarly if $(\mu_1, \nu_1) \in \mathfrak{h}^*$ is dominant integral with respect to $\mathfrak{b}$, $\mu_1 \in \mathfrak{h}_0^*$, $\nu_1 \in \mathfrak{a}_\mathbb{C}^*$, the corresponding irreducible finite-dimensional representation of G is denoted by F_{μ_1,ν_1}.

For a finite-dimensional representation (π, F) of G, denote by

$$
F^N = \{v \in F \mid \pi(n)v = v \quad \forall n \in N\}
$$

the space of N-fixed vectors. Similarly $F^{\overline{N}}$ denotes the space of $\overline{N}$-fixed vectors.

We have the following standard lemma [W1].

(2.1.4) **Lemma.** *Let (π, F) be an irreducible finite-dimensional representation of G, and (π^*, F^*) the contragradient representation. Then F^N and $(F^*)^{\overline{N}}$*

are irreducible representations of MA. Moreover, as representations of MA, they are contragradient to each other.

Thus in the notation of this section,

$$F^N_{\mu_1,\nu_1} \cong \sigma_{\mu_1} \otimes a^{\nu_1}$$

as MA-modules.

(2.1.5) Given an admissible representation (π, V) of G, there is a resulting representation of $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$ on V_0, the space of K-finite vectors, where K is a maximal compact subgroup of G. If $\mathscr{Z}(\mathfrak{g}_{\mathbb{C}})$ acts by scalars on V_0, the corresponding homomorphism

$$\chi : \mathscr{Z}(\mathfrak{g}_{\mathbb{C}}) \to \mathbb{C},$$

defined by

$$z \cdot x = \chi(z)x, \qquad x \in V_0, \ z \in \mathscr{Z}(\mathfrak{g}_{\mathbb{C}}),$$

is called the *infinitesimal character* of π.

More generally, if for some integer $d > 0$,

$$(z - \chi(z))^d x = 0 \quad \text{for all } x \in V_0, \ z \in \mathscr{Z}(\mathfrak{g}_{\mathbb{C}}),$$

we say π has the generalized infinitesimal character χ.

If $\chi = \chi_\lambda$, $\lambda \in C_G$ (see (2.1.2)), we shall also say π has the infinitesimal character λ (resp. generalized infinitesimal character λ).

We quote the following result from [Kn]. Let (π, V) be a Harish-Chandra module, i.e., a finitely generated admissible G-module. Then there exist linear functionals $\lambda_1, \dots, \lambda_l$ on $\mathfrak{h}$, $\mathscr{U}(\mathfrak{g}_{\mathbb{C}})$-invariant subspaces $V_1, \dots, V_l$ of V_0, and an integer $d > 0$ such that

(a) $\lambda_1, \dots, \lambda_l$ are mutually inequivalent under the Weyl group.

(b) $V_0 = V_1 \oplus \cdots \oplus V_l$.

(c) $(z - \chi_{\lambda_j}(z))^d$ acts as the zero operator in V_j for all $z \in \mathscr{Z}(\mathfrak{g}_{\mathbb{C}})$.

Thus, we have a canonically defined projection operator, denoted by P^G_λ, from the space of K-finite vectors of V to the subspace with the generalized infinitesimal character λ.

Similarly, if E is a Harish-Chandra module of M, we denote by P^M_μ the projection operator from the space of $K \cap M$ finite vectors of E to the subspace with the generalized infinitesimal character $\mu \in C_M$, the positive Weyl chamber specified by our choice of $\mathfrak{b}_0$ in $\mathfrak{m}_{\mathbb{C}}$.

2.2. **Two projection maps: P_N and $P_{\overline{N}}$.** From now on, we fix $F = F_{\mu_1,\nu_1}$, an irreducible finite-dimensional representation of G.

(2.2.1) **Lemma.** *We have the following direct sum decompositions:*

$$F = F^N \oplus \bar{\mathfrak{n}}F, \tag{2.2.2}$$

$$F^* = (F^*)^{\overline{N}} \oplus \mathfrak{n}F^* \quad (\textit{as } MA\textit{-modules}), \tag{2.2.3}$$

where $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ are Lie algebras of N and $\overline{N}$, respectively.

Proof. Use $\mathfrak{g}_{\mathbb{C}} = \bar{\mathfrak{n}} + \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$, the Poincare-Birkhoff-Witt theorem, and the irreducibility of F. We leave the details to the reader.

We denote by P_N and $P_{\overline{N}}$, the projections to F^N and $(F^*)^{\overline{N}}$, according to the decompositions (2.2.2) and (2.2.3), respectively. They are MA-homomorphisms.

We prove the following properties of P_N and $P_{\overline{N}}$.

(2.2.4) **Proposition.** (i) $P_N(\overline{n}v) = P_N(v)$, $\overline{n} \in \overline{N}$, $v \in F$; $P_{\overline{N}}(nv^*) = P_{\overline{N}}(v^*)$, $n \in N$, $v^* \in F^*$.

(ii) *P_N and $P_{\overline{N}}$ are adjoint to each other as elements of* $\mathrm{End}(F)$ *and* $\mathrm{End}(F^*)$.

Proof. (i) By definition, we have $P_N(Xv) = 0$, $X \in \overline{\mathfrak{n}}$, $v \in F$. From the well-known relationship between a Lie group and its Lie algebra, it follows that $P_N(\overline{n}v) = P_N(v)$, $\overline{n} \in \overline{N}$, $v \in F$.

(ii) Let

$$v = v_1 + v_2, \qquad v_1 \in F^N, \ v_2 \in \overline{\mathfrak{n}}F,$$
$$v^* = v_1^* + v_2^*, \qquad v_1^* \in (F^*)^{\overline{N}}, \ v_2^* \in \mathfrak{n}F^*.$$

Since $v_1^* \in (F^*)^{\overline{N}}$, we have $v_1^*(Xv) = -(Xv_1^*)(v) = 0$ for $X \in \overline{\mathfrak{n}}$, $v \in F$. Therefore $v_1^*(v_2) = 0$, and $v_2^*(v_1) = 0$ by a similar computation. Hence,

$$(P_{\overline{N}}v^*)(v) = v_1^*(v) = v_1^*(v_1),$$
$$v^*(P_N(v)) = v^*(v_1) = v_1^*(v_1).$$

2.3. **Definitions of four maps** T, S, U, V. Let $F = F_{\mu_1,\nu_1}$ be the irreducible representation of G such that $F^N = \sigma_{\mu_1} \otimes a^{\nu_1}$ as representations of MA. Let σ be an admissible representation of M with an infinitesimal character $\mu \in C_M$ (see §2.1). We shall write σ_μ instead of σ for the sake of notation.

Consider $I_{P,\mu,\nu} = \mathrm{Ind}_{MAN}^G(\sigma_\mu \otimes a^\nu \otimes 1)$. Let

$$\sigma_{\mu+\mu_1} \overset{\text{def}}{=} P_{\mu+\mu_1}^M(\sigma_\mu \otimes \sigma_{\mu_1}).$$

We recall here that $P_{\mu+\mu_1}^M$ is the projection operator onto the generalized infinitesimal character $\mu + \mu_1$. We caution the reader that σ_{μ_1} has the infinitesimal character $\mu_1 + \rho_M$ instead of μ_1, where ρ_M is the half sum of the positive $\mathfrak{h}_0$ roots in $\mathfrak{m}_{\mathbf{C}}$.

Let

$$I_{P,\mu+\mu_1,\nu+\nu_1} \overset{\text{def}}{=} \mathrm{Ind}_{MAN}^G(\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1).$$

We shall define below *four maps* T, S, U, V, which fit into the following two diagrams:

$$\text{(2.3.1)} \qquad \begin{array}{ccc} I_{P,\mu,\nu} \otimes F_{\mu_1,\nu_1} & \xrightarrow{J(\overline{P}:P,\mu,\nu)\otimes I} & I_{\overline{P},\mu,\nu} \otimes F_{\mu_1,\nu_1} \\ \Big\uparrow T & & \Big\downarrow S \\ I_{P,\mu+\mu_1,\nu+\nu_1} & \xrightarrow{J(\overline{P}:P,\mu+\mu_1,\nu+\nu_1)} & I_{\overline{P},\mu+\mu_1,\nu+\nu_1} \end{array}$$

$$\text{(2.3.2)} \qquad \begin{array}{ccc} I_{P,\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1} & \xrightarrow{J(\overline{P}:P,\mu+\mu_1,\nu+\nu_1)\otimes I} & I_{\overline{P},\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1} \\ \Big\downarrow U & & \Big\uparrow V \\ I_{P,\mu,\nu} & \xrightarrow{J(\overline{P}:P,\mu,\nu)} & I_{\overline{P},\mu,\nu} \end{array}$$

The reader is referred to §§1.1 and 1.2 for various notation used in the following.

We have the natural inclusion:

$$\sigma_{\mu+\mu_1} \hookrightarrow \sigma_\mu \otimes \sigma_{\mu_1}.$$

This will induce a natural inclusion of $\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1$ into

$$\begin{aligned}(\sigma_\mu \otimes \sigma_{\mu_1}) \otimes a^{\nu+\nu_1} \otimes 1 &= (\sigma_\mu \otimes a^\nu \otimes 1) \otimes (\sigma_{\mu_1} \otimes a^{\nu_1} \otimes 1)\\ &= (\sigma_\mu \otimes a^\nu \otimes 1) \otimes (F_{\mu_1,\nu_1})^N,\end{aligned}$$

but $(F_{\mu_1,\nu_1})^N$ is a MA-submodule of $(F_{\mu_1,\nu_1})|_P$, so

$$T_0 : \ \sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1 \hookrightarrow (\sigma_\mu \otimes a^\nu \otimes 1) \otimes (F_{\mu_1,\nu_1})|_P \tag{2.3.3}$$

as P-modules.

Therefore, we have a *natural* G-*map*

$$\begin{aligned} T = \beta(T_0)_G\colon \operatorname{Ind}_P^G(\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1) &\\ \xrightarrow{(T_0)_G} \operatorname{Ind}_P^G((\sigma_\mu \otimes a^\nu \otimes 1) \otimes (F_{\mu_1,\nu_1})|_P)&\\ \overset{\beta}{\cong} \operatorname{Ind}_P^G(\sigma_\mu \otimes a^\nu \otimes 1) \otimes F_{\mu_1,\nu_1}.&\end{aligned} \tag{2.3.4}$$

Since the image of $T_0 \subseteq (\sigma_\mu \otimes a^\nu \otimes 1) \otimes (F_{\mu_1,\nu_1})^N$, the above map T can be computed as follows:

$$T(f) = \sum_i f_i \otimes y_i, \qquad f \in I_{P,\mu+\mu_1,\nu+\nu_1}, \tag{2.3.5}$$

where $f_i(g) = f(g) \circ P_{\overline{N}}(g^{-1}y_i^*)$, with $\{y_i\}$ and $\{y_i^*\}$ dual bases of F and F^* (see (1.2.14) and Proposition (2.2.4)).

Let U be the following natural G-map:

$$\begin{aligned} U = (T_0^\dagger)_G\alpha\colon \operatorname{Ind}_P^G(\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1) \otimes F^*_{\mu+\mu_1}&\\ \overset{\alpha}{\cong} \operatorname{Ind}_P^G((\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1) \otimes F^*_{\mu_1,\nu_1})&\\ \xrightarrow{(T_0^\dagger)_G} \operatorname{Ind}_P^G(\sigma_\mu \otimes a^\nu \otimes 1).&\end{aligned} \tag{2.3.6}$$

We know by (1.2.11) that $U = T^\dagger$, i.e.,

$$U(f \otimes v^*) = T(f) \circ v^*. \tag{2.3.7}$$

By (1.2.13), we have

$$U(f \otimes v^*)(g) = f(g) \circ P_{\overline{N}}(g^{-1}v^*), \qquad f \in I_{P,\mu+\mu_1,\nu+\nu_1}. \tag{2.3.8}$$

Also, we have the following natural epimorphism:

$$\begin{aligned} &\sigma_\mu \otimes \sigma_{\mu_1} \twoheadrightarrow \sigma_{\mu+\mu_1},\\ &u \otimes u_1 \to P^M_{\mu+\mu_1}(u \otimes u_1), \qquad u \in \sigma_\mu,\ u_1 \in \sigma_{\mu_1}.\end{aligned}$$

This will induce a MA-map

$$\sigma_\mu \otimes a^\nu \otimes (F_{\mu_1,\nu_1})^N \twoheadrightarrow \sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1},$$

where $(F_{\mu_1,\nu_1})^N$ is a MA-quotient of F_{μ_1,ν_1} under the map

$$P_N :\ F_{\mu_1,\nu_1} \twoheadrightarrow (F_{\mu_1,\nu_1})^N.$$

So we have

$$\sigma_\mu \otimes a^\nu \otimes F_{\mu_1,\nu_1} \twoheadrightarrow \sigma_\mu \otimes a^\nu \otimes (F_{\mu_1,\nu_1})^N \twoheadrightarrow \sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1}.$$

Since the map P_N has the property $P_N(\overline{n}v) = P_N(v)$ for $\overline{n} \in \overline{N}$ (Proposition (2.2.4)), the composition of the above two epimorphisms yields an epimorphism

$$S_0 : (\sigma_\mu \otimes a^\nu \otimes 1) \otimes F_{\mu_1,\nu_1}|_{\overline{P}} \twoheadrightarrow \sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1 \tag{2.3.9}$$

as $\overline{P} = MA\overline{N}$-modules.

Therefore, we have a natural G-map

$$\begin{aligned} S = (S_0)_G\alpha \colon \operatorname{Ind}_{\overline{P}}^G(\sigma_\mu \otimes a^\nu \otimes 1) \otimes F_{\mu_1,\nu_1} & \\ \overset{\alpha}{\cong} \operatorname{Ind}_{\overline{P}}^G((\sigma_\mu \otimes a^\nu \otimes 1) \otimes F_{\mu_1,\nu_1}|_{\overline{P}}) & \\ \xrightarrow{(S_0)_G} \operatorname{Ind}_{\overline{P}}^G(\sigma_{\mu+\mu_1} \otimes a_{\nu+\nu_1} \otimes 1). & \end{aligned} \tag{2.3.10}$$

It can be computed as follows:

$$S(h \otimes v)(g) = P^M_{\mu+\mu_1}(h(g) \otimes P_N(g^{-1}v)), \qquad h \in I_{\overline{P},\mu,\nu}. \tag{2.3.11}$$

Let V be the following natural G-map,

$$\begin{aligned} V = \beta(S_0^\dagger)_G \colon \operatorname{Ind}_{\overline{P}}^G(\sigma_\mu \otimes a^\nu \otimes 1) & \\ \xrightarrow{(S_0^\dagger)_G} \operatorname{Ind}_{\overline{P}}^G(\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1 \otimes F^*_{\mu_1,\nu_1}|_{\overline{P}}) & \\ \overset{\beta}{\cong} \operatorname{Ind}_{\overline{P}}^G(\sigma_{\mu+\mu_1} \otimes a^{\nu+\nu_1} \otimes 1) \otimes F^*_{\mu_1,\nu_1}. & \end{aligned} \tag{2.3.12}$$

By (1.2.11), $S = V^\dagger$. Therefore,

$$V(h) = \sum S(h \otimes y_i) \otimes y_i^*, \tag{2.3.13}$$

or explicitly, by (2.3.11)

$$V(h) = \sum h_i \otimes y_i^*, \qquad h \in I_{\overline{P},\mu,\nu}, \tag{2.3.14}$$

where $h_i(g) = P^M_{\mu+\mu_1}(h(g) \otimes P_N(g^{-1}y_i))$.

(2.3.15) *Remark.* When $\mu_1 = 0$, i.e., M acts trivially on F^N, by a theorem of Helgason [He], F^N is one dimensional and F is spherical in the sense that it has a K-fixed vector, where K is a maximal compact subgroup of G, as before. The above definitions of T, S, U, V reduce to the formulas of T, S, U, V given in [VW].

2.4. **First commuting diagram.** Choose a constant c such that, if $\operatorname{Re}(\nu, \alpha) \geqslant c$ for $\alpha \in \Phi(P, A)$, then both integrals defining $J(\overline{P} : P, \mu, \nu)$ and $J(\overline{P} : P, \mu+\mu_1, \nu+\nu_1)$ converge absolutely (see the Introduction).

(2.4.1) **Proposition.** *If* $\operatorname{Re}(\nu, \alpha) \geqslant c$ *for* $\alpha \in \Phi(P, A)$, *then the following diagram commutes*:

$$\begin{array}{ccc} I_{P,\mu,\nu} \otimes F_{\mu_1,\nu_1} & \xrightarrow{J(\overline{P} : P, \mu, \nu) \otimes I} & I_{\overline{P},\mu,\nu} \otimes F_{\mu_1,\nu_1} \\ \uparrow{\scriptstyle T} & & {\scriptstyle S}\downarrow \\ I_{P,\mu+\mu_1,\nu+\nu_1} & \xrightarrow{J(\overline{P} : P, \mu+\mu_1, \nu+\nu_1)} & I_{\overline{P},\mu+\mu_1,\nu+\nu_1} \end{array} \tag{2.4.2}$$

Proof. Let $\{y_i\}$ and $\{y_i^*\}$ be dual bases of F_{μ_1,ν_1} and $F^*_{\mu_1,\nu_1}$, as before. Set $f_i(g) = f(g) \circ P_{\overline{N}}(g^{-1}y_i^*)$ for $f \in I_{P,\mu+\mu_1,\nu+\nu_1}$. Then, by (2.3.5),

$$T(f) = \sum_i f_i \otimes y_i.$$

Thus,

$$\begin{aligned} S(J(\overline{P}:P,\mu,\nu)\otimes I)Tf &= \sum_i S(J(\overline{P}:P,\mu,\nu)\otimes I)(f_i \otimes y_i) \\ &= \sum_i S(J(\overline{P}:P,\mu,\nu)f_i \otimes y_i). \end{aligned}$$

By the formula of S in (2.3.11), we have

$$\begin{aligned} &S(J(\overline{P}:P,\mu,\nu)f_i \otimes y_i)(g) \\ &\quad = P^M_{\mu+\mu_1}((J(\overline{P}:P,\mu,\nu)f_i)(g) \otimes P_N(g^{-1}y_i)) \\ &\quad = P^M_{\mu+\mu_1}\left(\int_{\overline{N}} f_i(g\overline{n})\,d\overline{n} \otimes P_N(g^{-1}y_i)\right) \\ &\quad = P^M_{\mu+\mu_1}\left(\int_{\overline{N}} (f(g\overline{n}) \circ P_{\overline{N}}((g\overline{n})^{-1}y_i^*)) \otimes P_N(g^{-1}y_i)\,d\overline{n}\right). \end{aligned} \tag{2.4.3}$$

Since

$$P_N(g^{-1}y_i) = P_N(\overline{n}^{-1}g^{-1}y_i) = P_N((g\overline{n})^{-1}y_i), \quad \text{(by Proposition (2.2.4))},$$

the expression inside the integral (2.4.3) is equal to

$$(f(g\overline{n}) \circ P_{\overline{N}}((g\overline{n})^{-1}y_i^*)) \otimes P_N((g\overline{n})^{-1}y_i).$$

Now

$$\begin{aligned} &\sum_i (f(g\overline{n}) \circ P_{\overline{N}}((g\overline{n})^{-1}y_i^*)) \otimes P_N((g\overline{n})^{-1}y_i) \\ &\quad = \sum_i (f(g\overline{n}) \circ P_{\overline{N}}(y_i^*)) \otimes P_N(y_i) = f(g\overline{n}) \end{aligned}$$

by Remark (1.1.6), since $f(g\overline{n}) \in \sigma_{\mu+\mu_1} = P^M_{\mu+\mu_1}(\sigma_\mu \otimes \sigma_{\mu_1}) \subseteq \sigma_\mu \otimes \sigma_{\mu_1}$, and the underlying space of σ_{μ_1} is F^N.

Therefore,

$$\begin{aligned} S(J(\overline{P}:P,\mu,\nu)\otimes I)Tf(g) &= P^M_{\mu+\mu_1}\left(\int_{\overline{N}} f(g\overline{n})\,d\overline{n}\right) = \int_{\overline{N}} f(g\overline{n})\,d\overline{n} \\ &= J(\overline{P}:P,\mu+\mu_1,\nu+\nu_1)f(g), \end{aligned}$$

which is the desired commutativity.

2.5. **Vogan's Lemma.** The following lemma is due to Vogan [V1].

(2.5.1) **Lemma.** *Let* $\mathfrak{g}$ *be a complex reductive Lie algebra,* $I \subseteq \mathscr{U}(\mathfrak{g})$ *any two-sided ideal, and let* $\mathscr{T}_I$ *denote the category of* $\mathscr{U}(\mathfrak{g})$*-modules* M *such that* $I \cdot M \equiv 0$. *Let* ϕ *be any natural construction which associates to each object* X *in* $\mathscr{T}_I$ *the following* $\mathscr{U}(\mathfrak{g})$*-module map:*

$$\phi_X : X \to X.$$

Then there is a unique element $\overline{Z} \in \mathscr{Z}(g)/I \cap \mathscr{Z}(g)$ *such that, for all* $X \in \mathscr{T}_I$,

$$\phi_X(x) = Z \cdot x \qquad (x \in X).$$

Here Z *is a representative of* $\overline{Z}$ *in* $\mathscr{Z}(\mathfrak{g})$. *In particular, if* $I \cap \mathscr{Z}(\mathfrak{g})$ *is maximal in* $\mathscr{Z}(\mathfrak{g})$, *then there is a constant* $c \in \mathbb{C}$ *such that*

$$\phi_X(x) = cx, \qquad x \in X.$$

(2.5.2) *Remark.* A formula for Z is also given in [V1], namely, $Z = \phi_{N_I}(\pi(1))$, where $N_I = \mathscr{U}(g)/I$ and π is the natural map from $\mathscr{U}(\mathfrak{g})$ to N_I.

From the above lemma, it follows

(2.5.3) **Proposition.** *Let* G *be a real reductive Lie group,* $\mathscr{T}_\lambda$ *the category of* $\mathscr{U}(\mathfrak{g}_\mathbb{C})$*-modules having the infinitesimal character* $\lambda \in \mathscr{C}_G$ ($\mathscr{C}_G$ *is a fixed positive Weyl chamber as in* §2.1), *and* F_Λ *the finite-dimensional irreducible representation of* G *with the highest weight* Λ. *Let* $X \in \mathscr{T}_\lambda$ *and define* $\phi : X \to X$ *by*

$$\phi(x) = \sum_i P^G_{\lambda+\Lambda}(x \otimes y_i) \circ y_i^*,$$

where $\{y_i\}$ *and* $\{y_i^*\}$ *are any dual bases of* F_Λ *and* F_Λ^*, *and* $\circ$ *is the contraction between* F_Λ *and* F_Λ^*. *Then there exists a scalar* $r^G_\lambda(\Lambda)$ *depending on* λ *and* Λ *such that* $\phi(x) = r^G_\lambda(\Lambda)x \ \forall x \in X$.

(2.5.4) *Remark.* Similarly, define

$$\psi(x) = \sum_i P^G_\lambda(x \otimes v_i^*) \circ v_i, \qquad x \in X \in \mathscr{T}_{\lambda+\Lambda}.$$

Then there exists a *scalar* $R^G_{\lambda+\Lambda}(-\Lambda)$ depending on λ and Λ such that $\psi(x) = R^G_{\lambda+\Lambda}(-\Lambda)x \ \forall x \in X$.

2.6. **Properties of** T, S, U, V. We refer the reader to §2.3 for the definitions of T, S, U, V. We first give a simple lemma about the induction functor.

(2.6.1) **Lemma.** Ind^G_{MAN} *and* $\mathrm{Ind}^G_{MA\overline{N}}$ *are exact functors.*

Proof. As an induction functor, it is always left exact. Since $G = KMAN$, restricting Ind^G_{MAN} to K gives an isomorphism.

Since

$$\mathrm{Ind}^G_{MAN}|K \cong \mathrm{Ind}^K_{M\cap K},$$

and $\mathrm{Ind}^K_{M\cap K}$ is right exact by the "unitary trick," we see that Ind^G_{MAN} is also right exact.

We observe that

(i) T_0 is injective.

(ii) S_0 is surjective.

The exactness of the two induction functors implies

(2.6.2) **Proposition.** T *is injective and* S *is surjective.*

Assume $r^M_\mu(\mu_1) \neq 0$, and let $\{w_j\}$ and $\{w_j^*\}$ be any dual bases of σ_{μ_1} and $\sigma^*_{\mu_1}$. Thus, by Proposition (2.5.3), we have

$$\frac{1}{r^M_\mu(\mu_1)} \sum_j P^M_{\mu+\mu_1}(u \otimes w_j) \circ w_j^* = u, \qquad u \in \sigma_\mu.$$

The above clearly implies

(iii) $\text{span}_{v^* \in \sigma_{\mu_1}^*}\{T_0(\sigma_{\mu+\mu_1}) \circ v^*\} = \text{span}_{v^* \in \sigma_{\mu_1}^*}\{(\sigma_{\mu+\mu_1}) \circ v^*\} = \sigma_\mu$.

(iv) If $u \in \sigma_\mu$ has the property that $S_0(u \otimes u_1) = P^M_{\mu+\mu_1}(u \otimes u_1) = 0 \; \forall u_1 \in \sigma_{\mu_1}$, then $u = 0$.

Again the exactness of the two induction functors implies

$$\text{span}_{v^* \in F^*_{\mu_1, \nu_1}} \{T(I_{P, \mu+\mu_1, \nu+\nu_1}) \circ v^*\} = I_{P, \mu, \nu}, \tag{2.6.3}$$

and if $S(h \otimes v) = 0 \;\; \forall v \in F$, then $h = 0$.

Since

$$U(f \otimes v^*) = T(f) \circ v^* \quad (\text{by } (2.3.7)),$$
$$V(h) = \sum S(h \otimes v_i) \otimes v_i^* \quad (\text{by } (2.3.13)),$$

we immediately have

(2.6.4) **Proposition.** *Assume* $r^M_\mu(\mu_1) \neq 0$. *Then* U *is surjective and* V *is injective.*

The proofs of the following two propositions can be copied almost word for word from [VW]. We give some details for the sake of completeness.

(2.6.5) **Proposition.** *There exists a nonzero complex valued polynomial* ϕ_1 *on* $\mathfrak{a}^*_{\mathbb{C}}$ *such that, if* $\phi_1(\nu) \neq 0$, *then*

$$T : I_{P, \mu+\mu_1, \nu+\nu_1} \to P^G_{\mu+\mu_1, \nu+\nu_1}(I_{P, \mu, \nu} \otimes F_{\mu_1, \nu_1})$$

and

$$S : P^G_{\mu+\mu_1, \nu+\nu_1}(I_{\overline{P}, \mu, \nu} \otimes F_{\mu_1, \nu_1}) \to I_{\overline{P}, \mu+\mu_1, \nu+\nu_1}$$

are linear bijections.

Proof. Let $F = F_1 \supset F_2 \supset \cdots \supset F_r \supset F_{r+1} = (0)$ be a Jordan-Holder series for $F = F_{\mu_1, \nu_1}$ as a P-module. We may assume that F_r is $\sigma_{\mu_1} \otimes a^\nu \otimes 1$ as a $P = MAN$-module. If V is an $(\mathfrak{m}, K \cap M)$-module and if $\lambda \in \mathfrak{a}^*_{\mathbb{C}}$, then we denote by V_λ the $(\mathfrak{p}, K \cap M)$-module V with $\mathfrak{n}$ acting trivially, $\mathfrak{m}$ acting as it did on V, and $\mathfrak{a}$ acting by λ. Then each F_i/F_{i+1} is of the form $(V_i)_{\lambda_i}$ with V_i an irreducible finite-dimensional $(\mathfrak{m}, K \cap M)$-module. Thus $I_{P, \mu, \nu} \otimes F$ has a composition series $I_{P, \mu, \nu} \otimes F = M_1 \supset M_2 \supset \cdots \supset M_r \supset M_{r+1} = (0)$ with $M_i/M_{i+1} \cong I_{P, \sigma_\mu \otimes (V_i)_{\nu+\lambda_i}}$. Now each $\sigma_\mu \otimes V_i$ has a direct sum decomposition into $(\mathfrak{m}, K \cap M)$-submodules V_{ij} each having a different (meaning inequivalent under the Weyl group) infinitesimal character (see §2.1). Thus, $I_{P, \sigma, \nu} \otimes F$ has a composition series with intermediate quotients $I_{P, V_{ij}, \nu+\lambda_i}$. Here λ_i is a weight of the action of $\mathfrak{a}$ on F and if $\lambda_i = \nu_1$, then $i = r$, and there is only one j with $V_{rj} \cong \sigma_{\mu+\mu_1}$ (cf. the definition of $\sigma_{\mu+\mu_1}$ in §2.3). Let μ_{ij} be the infinitesimal character parameter for V_{ij}. Then

$$\begin{aligned}
&\chi_{(\mu_{ij}, \nu+\lambda_i)}(C) - \chi_{(\mu+\mu_1, \nu+\nu_1)}(C) \\
&\quad = ((\mu_{ij}, \nu + \lambda_i), (\mu_{ij}, \nu + \lambda_i)) \\
&\qquad - ((\mu + \mu_1, \nu + \nu_1), (\mu + \mu_1, \nu + \nu_1)) \\
&\quad = (\mu_{ij}, \mu_{ij}) - (\mu + \mu_1, \mu + \mu_1) + (\lambda_i, \lambda_i) \\
&\qquad - (\nu_1, \nu_1) + 2(\nu, \lambda_i - \nu_1) = \phi_{ij}(\nu),
\end{aligned}$$

where C is the second order Casimir operator of G. It is obvious that $\phi_{ij}(\nu)$ is a nonzero polynomial of ν for $i < r$.

Set $\phi_1 = \prod_{\{i<r,j\}} \phi_{ij}$. Then if $\phi_1(\nu) \neq 0$, the only $I_{P, V_{ij}, \nu+\lambda_i}$ which can have the infinitesimal character $(\mu+\mu_1, \nu+\nu_1)$ is the one with $i = r$ and $V_{rj} \cong \sigma_{\mu+\mu_1}$, and so $P^G_{\mu+\mu_1, \nu+\nu_1}(I_{P,\mu,\nu} \otimes F) \cong I_{P,\mu+\mu_1,\nu+\nu_1}$. Similarly, if $\phi_1(\nu) \neq 0$, then $P^G_{\mu+\mu_1,\nu+\nu_1}(I_{\overline{P},\sigma,\nu} \otimes F) \cong I_{\overline{P},\mu+\mu_1,\nu+\nu_1}$. Since T is injective and S is surjective, the proposition now follows.

In view of Proposition (2.6.4), a similar argument as in above gives the following

(2.6.6) **Proposition.** *Assume $r^M_\mu(\mu_1) \neq 0$. Then there exists a nonzero complex valued polynomial ϕ_2 on $\mathfrak{a}^*_{\mathbb{C}}$ such that, if $\phi_2(\nu) \neq 0$, then*

$$U : P^G_{\mu,\nu}(I_{P,\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1}) \to I_{P,\mu,\nu}$$

and

$$V : I_{\overline{P},\mu,\nu} \to P^G_{\mu,\nu}(I_{\overline{P},\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1})$$

are linear bijections.

(2.6.7) *Remark.* $r^M_\mu(\mu_1) \neq 0$ if μ is a dominant integral regular infinitesimal character (see §2.8).

2.7. **Second commuting diagram: functional equations for the intertwining operators.** Again choose a constant c such that, if $\operatorname{Re}(\nu, \alpha) \geqslant c$ for $\alpha \in \Phi(P, A)$, then both integrals defining $J(\overline{P} : P, \mu, \nu)$ and $J(\overline{P} : P, \mu + \mu_1, \nu + \nu_1)$ converge absolutely.

Assume $r^M_\mu(\mu_1) \neq 0$. Our main result is the following

(2.7.1) **Theorem.** *If $\phi_1(\nu) \neq 0$, $\phi_2(\nu) \neq 0$ (see §2.6 for their definitions), $\operatorname{Re}(\nu, \alpha) \geqslant c$ for $\alpha \in \Phi(P, A)$, then the following diagram is commutative.*
(2.7.2)

$$\begin{array}{ccc}
P^G_{\mu,\nu}(I_{P,\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1}) & \xrightarrow{r^G_{\mu,\nu}(\mu_1,\nu_1)J(\mu+\mu_1,\nu+\nu_1)\otimes I} & P^G_{\mu,\nu}(I_{\overline{P},\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1}) \\
\Big\downarrow U & & \Big\uparrow V \\
I_{P,\mu,\nu} & \xrightarrow{J(\mu,\nu)} & I_{\overline{P},\mu,\nu}
\end{array}$$

Proof. Let $\{y_i\}$ and $\{y_i^*\}$ be dual bases of F_{μ_1,ν_1} and $F^*_{\mu_1,\nu_1}$, as before, and let $f = \sum_i f_i \otimes y_i^*$ be any element in $P^G_{\mu,\nu}(I_{P,\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1})$. Then

$$(J(\mu + \mu_1, \nu + \nu_1) \otimes I)\left(\sum_i f_i \otimes y_i^*\right) \in P^G_{\mu,\nu}(I_{\overline{P},\mu+\mu_1,\nu+\nu_1} \otimes F^*_{\mu_1,\nu_1}).$$

Since $r^M_\mu(\mu_1) \neq 0$ and $\phi_2(\nu) \neq 0$, V is a linear isomorphism by Proposition (2.6.6).

Therefore, there is a unique $h \in I_{\overline{P},\mu,\nu}$ such that

$$\begin{aligned}(J(\mu + \mu_1, \nu + \nu_1) \otimes I)\left(\sum_i f_i \otimes y_i^*\right) \\ = \sum_i J(\mu + \mu_1, \nu + \nu_1) f_i \otimes y_i^* = V(h).\end{aligned} \tag{2.7.3}$$

By (2.3.13),

$$V(h) = \sum_i S(h \otimes y_i) \otimes y_i^*,$$

so

$$\sum_i J(\mu + \mu_1, \nu + \nu_1) f_i \otimes y_i^* = \sum_i S(h \otimes y_i) \otimes y_i^*,$$

and hence

$$J(\mu + \mu_1, \nu + \nu_1) f_i = S(h \otimes y_i), \quad \text{for each } i.$$

By the first commuting diagram (2.4.2), namely,

$$S(J(\mu, \nu) \otimes I)T = J(\mu + \mu_1, \nu + \nu_1),$$

we obtain

$$\begin{aligned} S(J(\mu, \nu) \otimes I)Tf_i &= J(\mu + \mu_1, \nu + \nu_1) f_i = S(h \otimes y_i) \\ &= S[P^G_{\mu+\mu_1, \nu+\nu_1}(h \otimes y_i)]. \end{aligned}$$

Since $\phi_1(\nu) \neq 0$, S is a linear isomorphism by Proposition (2.6.5), and so

$$(J(\mu, \nu) \otimes I)Tf_i = P^G_{\mu+\mu_1, \nu+\nu_1}(h \otimes y_i).$$

Thus, by contracting with y_i^* and summing over i, we obtain by Proposition (2.5.3)

$$\sum_i ((J(\mu, \nu) \otimes I)Tf_i) \circ y_i^* = \sum_i P^G_{\mu+\mu_1, \nu+\nu_1}(h \otimes y_i) \circ y_i^* = r^G_{\mu,\nu}(\mu_1, \nu_1)h.$$

The left-hand side is equal to

$$\begin{aligned} \sum_i J(\mu, \nu)(Tf_i \circ y_i^*) &= \sum_i J(\mu, \nu) U(f_i \otimes y_i^*) \quad (\text{by } (2.3.7)) \\ &= J(\mu, \nu) U\left(\sum_i f_i \otimes y_i^*\right). \end{aligned}$$

Thus, we obtain

$$J(\mu, \nu) U\left(\sum_i f_i \otimes y_i^*\right) = r^G_{\mu,\nu}(\mu_1, \nu_1)h. \tag{2.7.4}$$

So,

$$\begin{aligned} &r^G_{\mu,\nu}(\mu_1, \nu_1)(J(\mu + \mu_1, \nu + \nu_1) \otimes I)\left(\sum_i f_i \otimes y_i^*\right) \\ &= r^G_{\mu,\nu}(\mu_1, \nu_1) V(h) \quad (\text{by } (2.7.3)) \\ &= VJ(\mu, \nu) U\left(\sum_i f_i \otimes y_i^*\right) \quad (\text{by } (2.7.4)), \end{aligned}$$

which is exactly the commutativity of the diagram (2.7.2).

(2.7.5) *Remark.* In [VW], a similar commutative diagram is proved with F an irreducible finite-dimensional spherical representation. Our proof is different from theirs.

(2.7.6) *Remark.* Since each representation present in diagram (2.7.2) is generically irreducible, one knows a priori that this diagram is commutative up to a multiple generically (Schur's Lemma).

To get this multiple, one forms $\sum_i VJ(\mu, \nu)UP^G_{\mu,\nu}(f\otimes y_i^*)\circ y_i$ and computes as follows:

$$\begin{aligned}
&\sum_i VJ(\mu, \nu)UP^G_{\mu,\nu}(f\otimes y_i^*)\circ y_i \\
&\quad = \sum_i VJ(\mu, \nu)U(f\otimes y_i^*)\circ y_i \\
&\quad = \sum_{i,j} S(J(\mu, \nu)U(f\otimes y_i^*)\otimes y_j)\otimes y_j^*\circ y_i \quad \text{(by (2.3.13))} \\
&\quad = \sum_i S(J(\mu, \nu)U(f\otimes y_i^*)\otimes y_i) \\
&\quad = \sum_i S(J(\mu, \nu)\otimes I)(U(f\otimes y_i^*)\otimes y_i) \\
&\quad = \sum_i S(J(\mu, \nu)\otimes I)Tf \quad \text{(since } U = T^\dagger, \text{ (see 2.3.7))} \\
&\quad = J(\mu+\mu_1, \nu+\nu_1)f \quad \text{(by the commutativity of diagram (2.4.2)).}
\end{aligned}$$

Thus,

$$\begin{aligned}
&R^G_{\mu+\mu_1,\nu+\nu_1}(-\mu_1, -\nu_1)\sum_i VJ(\mu, \nu)UP^G_{\mu,\nu}(f\otimes y_i^*)\circ y_i \\
&\quad = R^G_{\mu+\mu_1,\nu+\nu_1}(-\mu_1, -\nu_1)J(\mu+\mu_1, \nu+\nu_1)f \\
&\quad = \sum_i P^G_{\mu,\nu}(J(\mu+\mu_1, \nu+\nu_1)f\otimes y_i^*)\circ y_i \quad \text{(by Remark (2.5.4)).}
\end{aligned}$$

Combining the above identity with the commutativity of diagram (2.7.2), we have

(2.7.7) **Corollary.** *If* $r^M_\mu(\mu_1) \neq 0$, $\phi_1(\nu) \neq 0$, $\phi_2(\nu) \neq 0$, *and* $\mathrm{Re}(\nu, \alpha) \geqslant c$ *for* $\alpha \in \Phi(P, A)$, *then*

$$r^G_{\mu,\nu}(\mu_1, \nu_1)R^G_{\mu+\mu_1,\nu+\nu_1}(-\mu_1, -\nu_1) = 1.$$

2.8. **About the factor** $r^G_\lambda(\Lambda)$. Recall the definition of $r^G_\lambda(\Lambda)$ in §2.5.

(2.8.1) **Proposition** (Wallach). *If* λ *is a dominant integral regular infinitesimal character, then*

$$r_\lambda(\Lambda) = \frac{\prod_{\alpha>0}(\lambda+\Lambda, \alpha)}{\prod_{\alpha>0}(\lambda, \alpha)}.$$

Proof. Let ρ_G be the half sum of positive $\mathfrak{h}$ roots in $\mathfrak{g}_{\mathbb{C}}$, as usual. Let $F_{\lambda-\rho_G}$ be the irreducible finite-dimensional $\mathfrak{g}_{\mathbb{C}}$-module with highest weight $\lambda-\rho_G$, which exists by our assumption on λ. By Proposition (2.5.3), we have

$$\sum_i P^G_{\lambda+\Lambda}(m\otimes y_i)\circ y_i^* = r^G_\lambda(\Lambda)m, \qquad m\in F_{\lambda-\rho_G}.$$

Let $\{w_j\}$ and $\{w_j^*\}$ be dual bases of $F_{\lambda-\rho_G}$ and $F^*_{\lambda-\rho_G}$. In particular, the above identity gives

$$\sum_i P^G_{\lambda+\Lambda}(w_j \otimes y_i) \circ y_i^* = r_\lambda^G(\Lambda) w_j \quad \forall j.$$

Hence by contracting with w_j^* and summing over j, we obtain

$$(2.8.2)\quad \sum_j \sum_i P^G_{\lambda+\Lambda}(w_j \otimes y_i) \circ y_i^* \circ w_j^* = r_\lambda^G(\Lambda) \sum_j w_j \circ w_j^* = r_\lambda^G(\Lambda) \dim F_{\lambda-\rho_G}.$$

Since the left-hand side is independent of the choice of dual bases $\{w_j \otimes y_i\}$ and $\{w_j^* \otimes y_i^*\}$ of $F_{\lambda-\rho_G} \otimes F_\Lambda$ and $(F_{\lambda-\rho_G} \otimes F_\Lambda)^*$, we have

$$(2.8.2) = \text{trace of } P^G_{\lambda+\Lambda} = \dim P^G_{\lambda+\Lambda}(F_{\lambda-\rho_G} \otimes F_\Lambda) = \dim F_{\lambda+\Lambda-\rho_G}.$$

Thus,

$$r_\lambda^G(\Lambda) = \frac{\dim F_{\lambda+\Lambda-\rho_G}}{\dim F_{\lambda-\rho_G}} = \frac{\prod_{\alpha>0}(\lambda+\Lambda, \alpha)/\prod_{\alpha>0}(\rho_G, \alpha)}{\prod_{\alpha>0}(\lambda, \alpha)\prod_{\alpha>0}(\rho_G, \alpha)} = \frac{\prod_{\alpha>0}(\lambda+\Lambda, \alpha)}{\prod_{\alpha>0}(\lambda, \alpha)}$$

by the famous Weyl dimension formula.

Let $\mathscr{C}(\Lambda)$ be the region of λ such that if τ is a $\mathfrak{h}$-weight of F_Λ and $\tau \neq \Lambda$, then $\chi_{\lambda+\tau}(C) \neq \chi_{\lambda+\Lambda}(C)$. Recall here that C is the second order Casimir operator and $\chi_{\lambda+\tau}$ is the infinitesimal character associated to $\lambda+\tau \in \mathfrak{h}^*$ via the Harish-Chandra isomorphism (see §2.1). Thus $\mathscr{C}(\Lambda)$ is the complement in $\mathfrak{h}^*$ of a finite number of hyperplanes.

The proof of the following proposition is essentially given in [VW].

(2.8.3) **Proposition.** *$r_\lambda(\Lambda)$ is a rational function in $\mathscr{C}(\Lambda)$.*

Proof. Let $N_\lambda = \mathscr{U}(\mathfrak{g})/I_\lambda$, where I_λ is the two-sided ideal of $\mathscr{U}(\mathfrak{g}_\mathbb{C})$ generated by $z - \chi_\lambda(z)$, $z \in \mathscr{Z}(\mathfrak{g}_\mathbb{C})$. Then

$$N_\lambda \otimes F_\Lambda = \bigoplus_{\tau \in \pi(F_\Lambda)} P^G_{\lambda+\tau}(N_\lambda \otimes F_\Lambda),$$

where $\pi(F_\Lambda)$ is the set of $\mathfrak{h}$-weights of F_Λ (see [Ko]).

Let $r(\tau)$ be such that

$$(C - \chi_{\lambda+\tau}(C))^{r(\tau)} P^G_{\lambda+\tau}(N_\lambda \otimes F_\Lambda) = 0.$$

Set

$$U_\lambda = \prod_{\tau \in \pi(F) - \{\Lambda\}} (C - \chi_{\lambda+\tau}(C))^{r(\tau)},$$

$\widetilde{Z}_\lambda = \chi_{\lambda+\Lambda}(U_\lambda)^{-1} U_\lambda$ (it makes sense since $\chi_{\lambda+\tau}(C) \neq \chi_{\lambda+\Lambda}(C)$, for $\tau \neq \Lambda$). Then the projection of $N_\lambda \otimes F_\Lambda$ onto $P^G_{\lambda+\Lambda}(N_\lambda \otimes F_\Lambda)$ is given by the action of $\widetilde{Z}_\lambda$ on $N_\lambda \otimes F_\Lambda$.

Let

$$Z_\lambda = \sum \widetilde{Z}_\lambda(1 \otimes y_i) \circ y_i^*.$$

Then $r_\lambda(\Lambda) = \chi_\lambda(Z_\lambda)$ by Remark (2.5.2). It is clear that $r_\lambda(\Lambda)$ is rational in λ.

Let $\Phi(\mathfrak{b}, \mathfrak{h})$ be the set of positive $\mathfrak{h}$ roots in $\mathfrak{g}_\mathbb{C}$ specified by our choice of Borel subalgebra $\mathfrak{b}$. The above proposition clearly implies

(2.8.4) **Corollary.** *There exists a* $c > 0$ *such that* $r_\lambda(\Lambda)$ *is a rational function of* λ *in the region* $\{\lambda | \mathrm{Re}(\lambda, \alpha) \geqslant c \text{ for } \alpha \in \Phi(\mathfrak{b}, \mathfrak{h})\}$.

Combining Corollary (2.8.4) with Proposition (2.8.1), and observing that a nonzero rational function cannot have "half a lattice" as its zeros, we obtain

(2.8.5) **Proposition.**

$$r_\lambda(\Lambda) = \frac{\prod_{\alpha>0}(\lambda+\Lambda, \alpha)}{\prod_{\alpha>0}(\lambda, \alpha)}$$

if $\mathrm{Re}(\lambda, \alpha) \geqslant c$ *for* $\alpha \in \Phi(\mathfrak{b}, \mathfrak{h})$.

References

[He] S. Helgason, *Groups and geometric analysis*, Academic Press, Orlando, Fla., 1984.

[Hu] J. Humphreys, *Introduction to Lie algebras and representation theory*, Graduate Texts in Math., vol. 9, Springer-Verlag, New York, 1987.

[Kn] A. Knapp, *Representation theory of semisimple Lie groups: An overview based on examples*, Princeton Univ. Press, Princeton, N.J., 1986.

[Ko] B. Kostant, *On the tensor product of a finite and an infinite dimensional representation*, J. Funct. Anal. **20** (1975), 257–285.

[KS1] A. Knapp and E. Stein, *Intertwining operators for semisimple groups*, Ann. of Math. (2) **93** (1971), 489–578.

[KS2] ——, *Intertwining operators for semisimple groups.* II, Invent. Math. **60** (1980), 9–84.

[Sl] S. Lang, $Sl_2(\mathbb{R})$, Graduate Texts in Math., vol. 105, Springer-Verlag, 1975.

[V1] D. Vogan, Jr., *Representations of real reductive Lie groups*, Birkhäuser, Boston, Mass., 1981.

[VW] D. Vogan, Jr. and N. Wallach, *Intertwining operators for real reductive groups*, Adv. Math. **82** (1990), 203–243.

[W1] N. Wallach, *Harmonic analysis on homogeneous spaces*, Marcel Dekker, New York, 1973.

[Zh] C. Zhu, *Two topics in harmonic analysis on reductive groups*, Thesis, Yale University, 1990.

[Zu] G. Zuckerman, *Tensor products of finite and infinite dimensional representations of semisimple Lie groups*, Ann. of Math. (2) **106** (1977), 295–308.

Department of Mathematics, University of Maryland, College Park, Maryland 20742

Current address: Department of Mathematics, National University of Singapore, Kent Ridge, Singapore 0511

E-mail address: matzhucb@nuscc.nus.sg